NOTES

Honey, Where Should We Sit?

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There are times when, in their haste to solve a particular problem, students (and their instructors) miss an opportunity to notice some interesting mathematics. For example, when calculus students are introduced to the derivatives of inverse trigonometric functions, they frequently run across a classic problem that goes something like this:

There is a 6-foot tall picture on a wall, 2 feet above your eye level. How far away should you sit (on the level floor) in order to maximize the vertical viewing angle θ? (See FIGURE 1.)

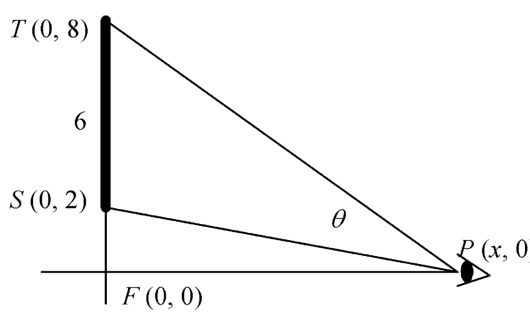

Figure 1 Find where θ is a maximum

This problem can be solved using the standard calculus technique for maximization. First, on the coordinate plane, we could set the top and bottom of the picture at $T(0, 8)$ and $S(0,\ 2)$, respectively. Then it is easy to show that if your eye is at a point $P(x, 0)$ on the positive x-axis, the viewing angle would be $\theta = \tan^{-1}(8/x) - \tan^{-1}(2/x)$. From the derivative,

$$\frac{d\theta}{dx} = \frac{6(16 - x^2)}{(x^2 + 8^2)(x^2 + 2^2)},$$

you can easily show that the only critical number for $x > 0$ occurs at $x = 4$. Finally, (the part that many students like to skip) the first or second derivative test can provide arguments that θ must be an absolute maximum at $P(4, 0)$.

At this point, many calculus students declare that the greatest viewing angle occurs 4 feet from the wall, express some relief and gratitude for having solved the problem, and move on to the next assignment. In doing so, unfortunately, they miss some fascinating geometry. Notice that, if we let F represent the origin, then at the point P of maximum θ, $PF/FS = 2 = TF/PF$ (FIGURE 2). This makes $\triangle PFS$ and

$\triangle TFP$ *similar* right triangles. Thus, the viewing angle is largest at the point P where $\angle FPS \cong \angle FTP$!

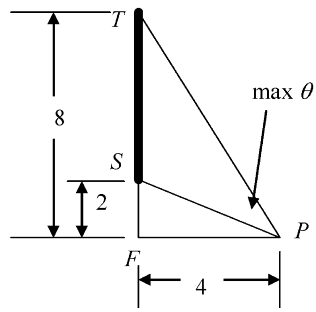

Figure 2 Similar triangles $\triangle PFS$ and $\triangle TFP$

So now a mathematician starts to wonder: is this result just a coincidence (if there is such a thing as a mathematical coincidence)? What if we change the y-coordinates of S and T? How about if, instead of being level, the floor were slanted and P were on a line $y = mx$? (Stewart gives a numerical approach to a variation of this problem [**1**, p. 478].)

Curiously enough, even in these cases the answer is that the viewing angle is a maximum where $\angle FPS \cong \angle FTP$. (This could be a good assignment for a bright student.) In fact, we can generalize even further and consider the case where the floor is curved rather than straight. The result is the following:

THEOREM. *Let $S(0, a)$ and $T(0, b)$ be points on the y-axis with $a < b$, and let $y = f(x)$ be a continuous function on* $[0, \infty)$ *and, without loss of generality, $f(0) < a$. Then there is point $P(x, f(x))$, $x > 0$, on the graph of f such that the measure of $\angle TPS$ is a maximum. Furthermore, if f is differentiable at P, then $\angle FPS \cong \angle FTP$, where F is the point where the tangent to $f(x)$ at P intersects the y-axis* (FIGURE 3).

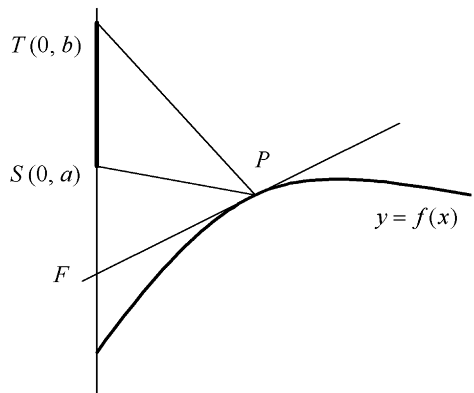

Figure 3 The generalized case

Note: In the original problem P is on the x-axis $y = 0$, and in the variation P is on the line $y = mx$. Both times, the point F is given as the origin. This notation is consistent with our generalized property since, in those cases, the tangent line to the graph of $y = f(x)$, which is simply the graph itself, intersects the y-axis at (0, 0). Also, when we refer to a maximum θ, or θ being maximized, we shall implicitly restrict ourselves to the domain $(0, \infty)$.

Proof. The property that $\angle FPS \cong \angle FTP$ at maximum θ can be proved using standard calculus. Suppose f is differentiable at the maximum angle. We will assume for the time being that a greatest θ exists. It is straightforward to show that, if point P has coordinates $(x, f(x))$, then $\angle TPS$ has measure

$$\theta = \tan^{-1}\left(\frac{b - f(x)}{x}\right) + \tan^{-1}\left(\frac{f(x) - a}{x}\right).$$

Differentiating and simplifying, we can see that

$$\frac{d\theta}{dx} = (a-b)\frac{\left[x^2 + (xf'(x))^2\right] - [a - (f(x) - xf'(x))][b - (f(x) - xf'(x))]}{[x^2 + (b - f(x))^2][x^2 + (a - f(x))^2]}.$$

Since the denominator involves products of sums of perfect squares, and since $f(0)$ is neither a nor b, we can see that the denominator is never zero; hence, $d\theta/dx$ is never undefined. It follows that at the maximum, the derivative must be zero. At this point then,

$$x^2 + (xf'(x))^2 = [a - (f(x) - xf'(x))]\,[b - (f(x) - xf'(x))]. \tag{1}$$

All we need to do is interpret this in terms of lengths. The slope of the tangent to $f(x)$ at P is $f'(x)$. If we follow the tangent line back to the y-axis, we see that F has coordinates $(0, f(x) - xf'(x))$, as in FIGURE 4.

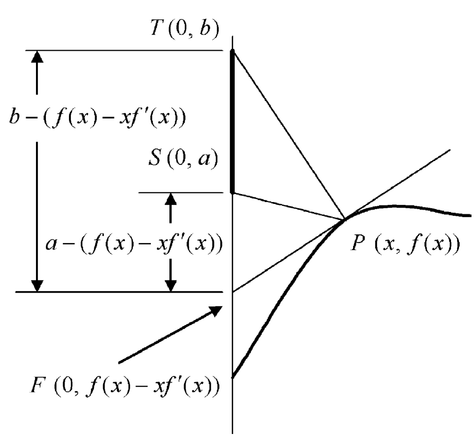

Figure 4 Where the tangent hits the y-axis

From (1), we see that $PF^2 = SF \cdot TF$, that is,

$$\frac{PF}{SF} = \frac{TF}{PF}.$$

Since they share a common angle and have two pairs of proportional sides, it follows that $\triangle SFP$ and $\triangle PFT$ are similar triangles. Therefore, we can conclude that $\angle FPS \cong \angle FTP$ when P is chosen to make $\angle TPS$ largest. ∎

Geometric approach Now we turn to some more general questions: Assuming f is continuous, not necessarily differentiable, on $[0, \infty)$, are we guaranteed that there is a point P where the viewing angle is greatest? If there is such a point P, is it necessarily unique or might the maximum angle occur at more than one point on the graph? We can answer these questions by taking a different approach to the problem. Let's leave calculus and its potentially messy computations and turn instead to geometry (with just a pinch of topology).

Recall that, in a circle, the measure of an inscribed angle is one-half that of the intercepted arc [**3**]. A corollary of this property is that every inscribed angle that intercepts the same arc has the same measure. Conversely, given fixed points T and S and an angle θ, the set of all points Q on one side of $\overline{ST}$ satisfying $m(\angle SQT) = \theta$ is a portion of a circle passing through S and T.

Now let's return to our problem. Again, we let S and T represent the top and bottom of our picture. For a fixed positive measure c, consider the set of points Q on the right half-plane such that $m(\angle SQT) = c$. From our discussion above, we can easily see that this level curve is the right-hand portion of a circle passing through S and T (FIGURE 5).

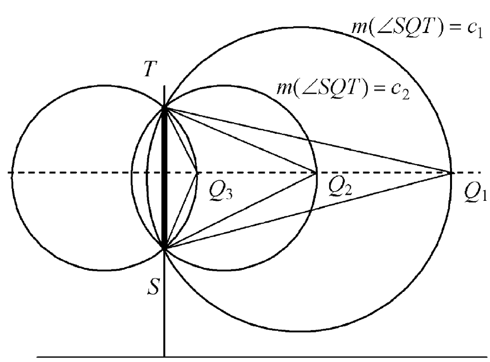

Figure 5 Level curves of constant angles

Moreover, the smaller the value of c, the farther the center of the circle is to the right. For instance, if Q_1, Q_2, and Q_3 are placed on the perpendicular bisector of $\overline{ST}$ as shown in FIGURE 5, it is easy to see that $m(\angle SQ_1T) < m(\angle SQ_2T) < m(\angle SQ_3T)$. Also notice that the regions bounded by $\overline{ST}$ and these circular curves are nested: If $0 < c_1 < c_2$, then the region bounded by $\overline{ST}$ and the curve $m(\angle SQT) = c_2$ is contained in the region bounded by $\overline{ST}$ and $m(\angle SQT) = c_1$.

Now we can answer the questions we posed earlier. Must there be a point P along the graph of $y = f(x)$ at which $m(\angle SPT)$ is a maximum? If so, where is P? The answer to the second question is that P occurs where $y = f(x)$ intersects the circular arc $m(\angle SQT) = c$ for the largest value of c, that is, the leftmost curve $m(\angle SQT) = c$ (FIGURE 6). It is probably obvious that there must be such a point; however, to be safe, we could turn to a little topology. (If this result is obvious, feel free to skip the next paragraph.)

Let G represent the graph of $y = f(x)$. For each positive c, let D_c be the closed bounded region in the right closed half-plane bounded by $\overline{ST}$ and the arc $m(\angle SQT) = c$. Then define G_c to be the intersection of G with D_c. Now consider the nonempty collection $A = \{G_c : G_c \neq \emptyset\}$ of nonempty intersections of G with the sets D_c. The continuity of f implies that G is closed; hence, each G_c is compact. Furthermore, since the D_cs are nested, it follows that the G_cs satisfy the finite intersection

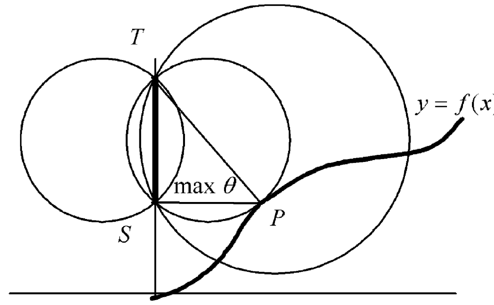

Figure 6 Where θ is maximized

property [**2**]. Therefore, $\bigcap_{G_c \in A} G_c \neq \emptyset$ and $m(\angle SPT)$ is a maximum at any point P in $\bigcap_{G_c \in A} G_c$.

We can see that this result is consistent with our earlier findings about similar triangles. If the tangent to the circle at P intersects the y-axis at F (FIGURE 7) then, since $\angle SPF$ and $\angle PTF$ intercept the same arc, they are congruent. Consequently, $\triangle SFP$and$\triangle PFT$ are similar triangles and $PF/SF = TF/PF$, as before.

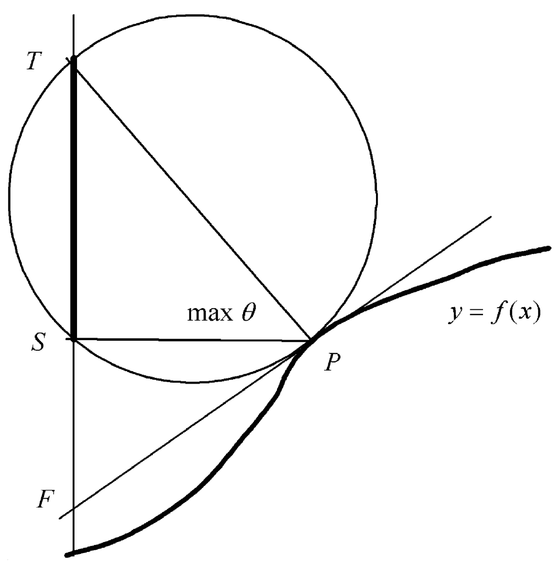

Figure 7 Similar triangles in the general case

This geometric approach allowed us to see, without ugly computations, that there must be a point P on G such that the viewing angle, $m(\angle SPT)$, is maximized. Furthermore, an easy construction allows us to show that, depending upon G, this point of greatest angle may occur at more than one point (FIGURE 8a). In fact, if G moves along a section of one such circular arc, there would be an infinite number of such points (FIGURE 8b).

We now address one final question: How do we construct such a point P? As we showed earlier, sometimes you can find P using possibly cumbersome calculus computations. In the special cases where the graph G is a line, however, we can use the geometry of the situation to physically construct the point of maximum angle using a compass and straightedge. In these situations the smallest circle through S and T that

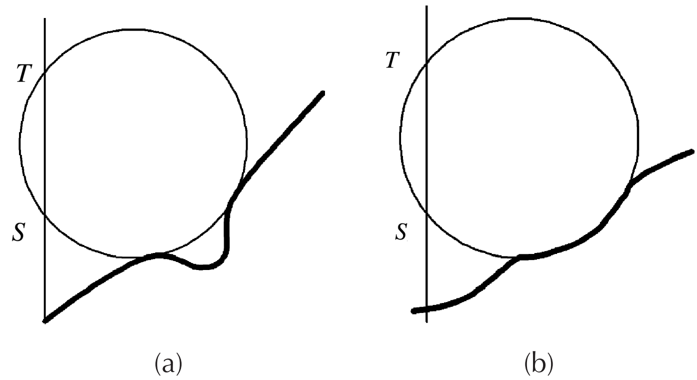

Figure 8 Cases where θ is maximized at multiple points

intersects G must be tangent to G at that point. Thus, all we need to do is find this tangent circle and determine the point P of tangency.

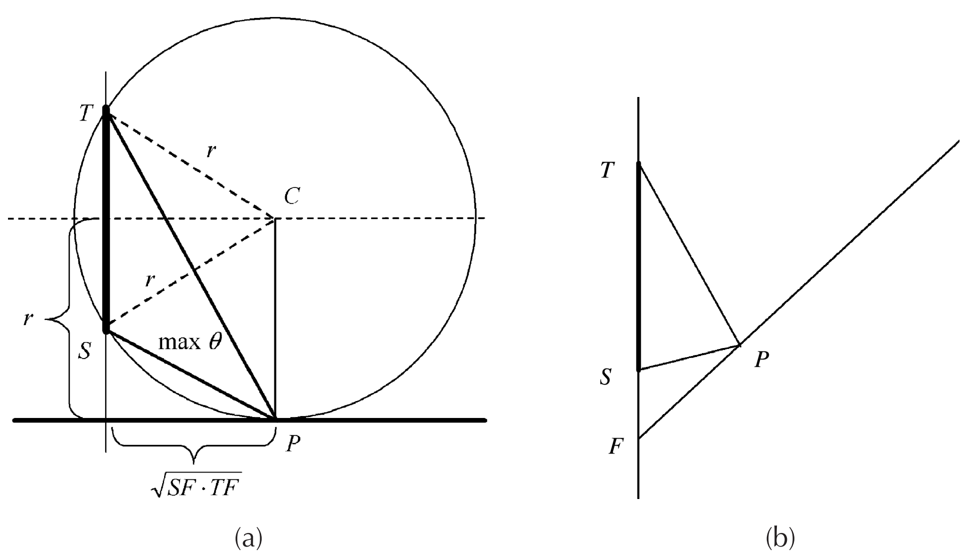

Figure 9 (a) Constructing the point of maximum θ (b) Constructing the slant line solution

This task is especially easy if G is a horizontal line (FIGURE 9a). In this situation, the one we started with, the smallest circle through S and T that intersects G must be tangent to G at that point. Thus, all we need to do is find this tangent circle and determine the point P of tangency. First we find the distance r from the perpendicular bisector of $\overline{ST}$ to G. Next we locate the point C on the right side of this perpendicular bisector that is r units from both S and T. The maximum angle then occurs at the foot P of the perpendicular from C to G. Notice that, from our previous discussion, $PF/SF = TF/PF$; hence, $PF = \sqrt{SF \cdot TF}$, so PF is the geometric mean of SF and TF.

Now that we've constructed the solution for a horizontal line, the solution for the slant line situation becomes easy. At the point of greatest angle measure, we still have the similar triangles, so the distance from P to F is still $PF = \sqrt{SF \cdot TF}$. We constructed this distance in the horizontal line case. All we need to is to construct a circle with center F and radius $\sqrt{SF \cdot TF}$. The desired point P is the intersection of this circle and the slant line (FIGURE 9b).

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2. James R. Munkres, *Topology, A First Course*, Prentice-Hall, Englewood Cliffs, NJ, 1975.
3. James R. Smart, *Modern Geometries*, 5th ed., Brooks/Cole, Pacific Grove, CA, 1998.